Variational and asymptotic analysis of a viscous fluid-3D thin plate interaction problem

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Description of the physical problem

We consider that

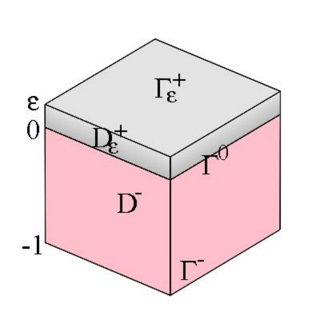
- the interaction problem takes place in an infinite layer $\mathbb{R}^2 \times (-1, \varepsilon)$,
- it is 1-periodic with respect to x_1, x_2.

The periodicity domains for the fluid and for the elastic layer respectively are

$$\begin{cases} D^- = (0,1) \times (0,1) \times (-1,0), \\ D_\varepsilon^+ = (0,1) \times (0,1) \times (0,\varepsilon) \end{cases}$$

The boundaries which do not correspond to the periodicity conditions are

$$\begin{cases} \Gamma^- = \{(x_1, x_2, -1)/(x_1, x_2) \in (0,1)^2\}, \\ \Gamma^0 = \{(x_1, x_2, 0)/(x_1, x_2) \in (0,1)^2\}, \\ \Gamma_\varepsilon^+ = \{(x_1, x_2, \varepsilon)/(x_1, x_2) \in (0,1)^2\}. \end{cases}$$



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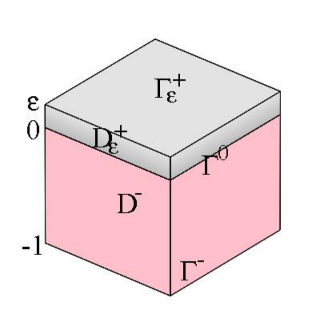

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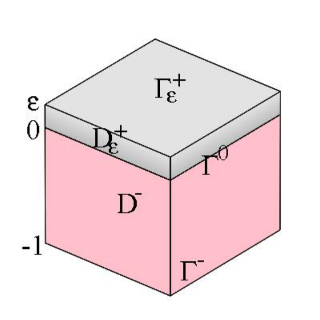

Properties for the data

Elastic medium

Density $\varepsilon^{-1}\rho_+(\xi_3)$, $\exists\,\alpha,\beta>0:\ \alpha\le\rho_+(\xi_3)\le\beta$,
matrix-valued coefficients $\varepsilon^{-3}A_{ij}(\xi_3)$, $i,j\in\{1,2,3\}$,
Young's modulus $\varepsilon^{-3}E(\xi_3)$, $E=\mathcal{O}(1)$,
Poisson's coefficient $\nu(\xi_3)$,
with $\xi_3=\dfrac{x_3}{\varepsilon}$.

Viscous fluid

Density ρ_- and viscosity $\tilde{\nu}$.

The matrices $A_{ij}=(a_{ij}^{kl})_{1\le k,l\le 3}$ are defined by

$$a_{ij}^{kl}=\frac{E}{2(1+\nu)}\left(\frac{2\nu}{1-2\nu}\delta_{ik}\delta_{jl}+\delta_{ij}\delta_{kl}+\delta_{il}\delta_{jk}\right)\text{ with the properties:}$$

(i) $a_{ij}^{kl}(\xi_3)=a_{kj}^{il}(\xi_3)=a_{ji}^{lk}(\xi_3),\ \forall\, i,j,k,l\in\{1,2,3\},\ \forall\xi_3\in[0,1]$,

$$\text{(ii) }\exists\,\kappa>0:\quad \sum_{i,j,k,l=1}^{3} a_{ij}^{kl}(\xi_3)\eta_j^l\eta_i^k\ge\kappa\sum_{j,l=1}^{3}(\eta_j^l)^2,\ \forall\xi_3\in[0,1],$$

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Motivation

Interaction between

- blood in the vessels and vascular wall,
- Earth crust and Earth mantle.

G. P. Panasenko and R. Stavre.
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SIAM J. Appl. Dyn. Sys., 2 (3) (2003) 431–463.

Description of the physical problem (continuation)

The notations for the velocity strain tensor and the linearized strain tensor are

$$D(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^T),\ \mathcal{E}(\varphi) = \frac{1}{2}\left(\nabla \varphi + (\nabla \varphi)^T\right). \tag{1}$$

We also know the forces $\mathbf{g}$ and $\mathbf{f}$ which act on the elastic medium and on the fluid, respectively, with the regularity

$$\begin{cases} \rho_+,\ a_{ij}^{kl} \in L^\infty(0,1),\ i,j,k,l \in \{1,2,3\}, \\ \mathbf{g} \in H^1(0,T;(L^2(D_\varepsilon^+))^3),\ \mathbf{f} \in H^1(0,T;(L^2(D^-))^3), \\ \mathbf{g},\mathbf{f}\ \ 1-\text{periodic in } x_1, x_2. \end{cases} \tag{2}$$

The unknowns

- $\mathbf{u}_\varepsilon$, representing the displacement of the elastic medium,
- $\mathbf{v}_\varepsilon$, p_ε, representing the velocity and the pressure of the viscous fluid, respectively.

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Statement of the problem

We consider the following coupled system which describes the fluid-elastic layer interaction problem in $\mathbb{R}^2 \times (-1, \varepsilon) \times (0, T)$, with T a positive given constant

$$\begin{cases} \varepsilon^{-1}\rho_+ \dfrac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - \varepsilon^{-3} \displaystyle\sum_{i,j=1}^{3} \dfrac{\partial}{\partial x_i}\left(A_{ij}\dfrac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) = \varepsilon^{-1}\mathbf{g} \text{ in } D_\varepsilon^+ \times (0,T), \\ \rho_- \dfrac{\partial \mathbf{v}_\varepsilon}{\partial t} - 2\tilde{\nu}\,\mathrm{div}\,(D(\mathbf{v}_\varepsilon)) + \nabla p_\varepsilon = \mathbf{f},\ \mathrm{div}\ \mathbf{v}_\varepsilon = 0 \text{ in } D^- \times (0,T), \\ \displaystyle\sum_{j=1}^{3} A_{3j}\dfrac{\partial \mathbf{u}_\varepsilon}{\partial x_j} = \mathbf{0} \text{ on } \Gamma_\varepsilon^+ \times (0,T), \\ \mathbf{v}_\varepsilon = \mathbf{0} \text{ on } \Gamma^- \times (0,T), \\ \mathbf{v}_\varepsilon = \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t},\ -p_\varepsilon \mathbf{e}_3 + 2\tilde{\nu} D(\mathbf{v}_\varepsilon)\mathbf{e}_3 = \varepsilon^{-3}\displaystyle\sum_{j=1}^{3} A_{3j}\dfrac{\partial \mathbf{u}_\varepsilon}{\partial x_j} \text{ on } \Gamma^0 \times (0,T), \\ \mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon \quad 1-\text{periodic in } x_1, x_2, \\ \mathbf{u}_\varepsilon(0) = \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t}(0) = \mathbf{0} \text{ in } D_\varepsilon^+, \mathbf{v}_\varepsilon(0) = \mathbf{0} \text{ in } D^-. \end{cases} \tag{3}$$

Variational analysis of the problem

Spaces

We define the spaces which allow us to introduce and study the coupled problem (3) from a variational viewpoint:

$$\begin{cases} U=\{\varphi \in (H^1(D_\varepsilon^+))^3 / \int_{(0,1)^2} \varphi_3(x_1,x_2,0)\,\mathrm{d}x_1\mathrm{d}x_2 = 0,\ \varphi\ 1-\text{periodic in } x_1,x_2\}, \\ V=\{\omega\in(H^1(D^-))^3/\operatorname{div}\omega = 0,\ \omega = \mathbf{0} \text{ on } \Gamma^-,\ \omega\ 1-\text{periodic in } x_1,x_2\}. \end{cases} \tag{4}$$

We also define the auxiliary spaces

$$\begin{cases} V_0 = \{\omega \in V/\ \omega = \mathbf{0} \text{ on } \Gamma^0\}, \\ S = \{(\varphi,\omega) \in U \times V/\varphi = \omega \text{ on } \Gamma^0\}. \end{cases} \tag{5}$$

Finally, we define the spaces for the unknowns $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$

$$\begin{cases} H_u = \{\varphi \in H^1(0,T;U)/\ \dfrac{\partial^2\varphi}{\partial t^2} \in L^2(0,T;U')\}, \\ H_v = \{\omega \in L^2(0,T;V)/\ \dfrac{\partial\omega}{\partial t} \in L^2(0,T;V')\}. \end{cases} \tag{6}$$

Variational analysis of the problem

Variational formulation

We consider the following variational problem

$$
\begin{cases}
\text{Find } (\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon) \in H_u \times H_v \text{ such that} \\
\varepsilon^{-1} \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D_\varepsilon^+} \rho_+ \left(\frac{x_3}{\varepsilon}\right) \frac{\partial \mathbf{u}_\varepsilon(\mathbf{t})}{\partial t} \cdot \boldsymbol{\varphi} + \varepsilon^{-3} \int_{D_\varepsilon^+} \sum_{i,j=1}^{3} A_{ij} \left(\frac{x_3}{\varepsilon}\right) \frac{\partial \mathbf{u}_\varepsilon(\mathbf{t})}{\partial x_j} \cdot \frac{\partial \boldsymbol{\varphi}}{\partial x_i} \\
\quad + \rho_- \dfrac{\mathrm{d}}{\mathrm{d}t} \displaystyle\int_{D^-} \mathbf{v}_\varepsilon(t) \cdot \boldsymbol{\omega} + 2\tilde{\nu} \int_{D^-} D(\mathbf{v}_\varepsilon(t)) : D(\boldsymbol{\omega}) \\
\quad = \varepsilon^{-1} \displaystyle\int_{D_\varepsilon^+} \mathbf{g}(t) \cdot \boldsymbol{\varphi} + \int_{D^-} \mathbf{f}(t) \cdot \boldsymbol{\omega} \quad \forall (\boldsymbol{\varphi}, \boldsymbol{\omega}) \in S, \ \text{a.e. in } (0, T), \\
\mathbf{v}_\varepsilon = \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t} \ \text{a.e. on } \Gamma^0 \times (0, T), \\
\mathbf{u}_\varepsilon(0) = \dfrac{\partial \mathbf{u}_\varepsilon}{\partial t}(0) = \mathbf{0} \ \text{a.e. in } D_\varepsilon^+, \ \mathbf{v}_\varepsilon(0) = \mathbf{0} \ \text{a.e. in } D^-.
\end{cases}
\tag{7}
$$

Variational analysis of the problem

Galerkin's method

By means of the sequences $\{\boldsymbol{\omega}_k\}_{k\in\mathbb{N}} \subset V_0$ and $\{(\boldsymbol{\varphi}_l, \boldsymbol{\psi}_l)\}_{l\in\mathbb{N}} \subset S$, we define the approximations for the displacement and for the velocity by

$$\begin{cases} \mathbf{u}_n(x,t) = \displaystyle\sum_{l=1}^{n} b_l(t)\boldsymbol{\varphi}_l(x_1,x_2,x_3), \text{ for } (x,t) \in D_\varepsilon^+ \times (0,T), \\ \mathbf{v}_n^m(x,t) = \displaystyle\sum_{k=1}^{m} a_k(t)\boldsymbol{\omega}_k(x_1,x_2,x_3) + \sum_{l=1}^{n} b_l'(t)\boldsymbol{\psi}_l(x_1,x_2,x_3), \text{ for } (x,t) \in D^- \times (0,T). \end{cases} \tag{8}$$

The unknowns of the previous definitions, the functions $a_k, b_l : [0,T] \mapsto \mathbb{R}$, are determined from the problem (the variational problem (7) written for the test functions $\{(\boldsymbol{\varphi}_l, \boldsymbol{\psi}_l)\}_{l\in\mathbb{N}}$ with $\boldsymbol{\psi}_l = \boldsymbol{\varphi}_l$ on Γ^0 and the variational formulation for the equation $(3)_2$ with the test function $\{\boldsymbol{\omega}_k\}_{k\in\mathbb{N}}$ vanishing on Γ^0).

Variational analysis of the problem (results)

Theorem 1

The problem (7) has a unique solution, $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon)$. For the data with the regularity given by (2) we obtain for the unknowns the properties

$$\frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} \in L^\infty(0, T; (L^2(D_\varepsilon^+))^3), \ \frac{\partial \mathbf{v}_\varepsilon}{\partial t} \in L^\infty(0, T; (L^2(D^-))^3). \tag{9}$$

Theorem 2

Let $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon)$ be the unique solution of (7). Then we have the supplementary regularity

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right) \in L^\infty(0, T; (L^2(D_\varepsilon^+))^3), \ \mathbf{v}_\varepsilon \in L^\infty(0, T; (H^2(D^-))^3) \tag{10}$$

and there exists a unique function $p_\varepsilon \in L^2(0, T; H^1(D^-))$ such that the triplet $(\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon, p_\varepsilon)$ satisfies (3) in a classical sense. Moreover, the 1-periodicity in x_1,x_2 of each unknown function corresponds to its regularity stated above.

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Asymptotic analysis of the problem

Hypothesis

For the asymptotic approach of the problem we need further regularity for the data. We suppose that

(H1) ρ_+ and a_{ij}^{kl} are piecewise smooth functions on $[0,1]$;

(H2) the function $\mathbf{g}$ is independent of x_3
and $\mathbf{g} \in C^\infty([0,\, T], (C^\infty_{per}([0,1]))^3)$;

(H3) the function $\mathbf{f}$ is C^∞ 1-periodic in x_1, x_2
and $\dfrac{\partial^l \mathbf{f}}{\partial x_{i_1} \dots \partial x_{i_l}} \in C^\infty([0,\, T], (L^2(D^-))^3)$, for any $l \in \mathbb{N}$,
$i = (i_1, \dots, i_l)$, $i_j \in \{1,2\}$, $|i| = l$;

(H4)
$\exists \tau_0 < T$ such that $\mathbf{f} = \mathbf{0}$ in $D^- \times [0, \tau_0]$, $\mathbf{g} = \mathbf{0}$ in $[0,1] \times [0, \tau_0]$.

Asymptotic analysis of the problem

Ideas

We introduce the following notation for the left hand side of $(3)_1$

$$P_\varepsilon \mathbf{u}_\varepsilon = \varepsilon^{-1}\rho_+\left(\frac{x_3}{\varepsilon}\right)\frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - \varepsilon^{-3}\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(A_{ij}\left(\frac{x_3}{\varepsilon}\right)\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right). \tag{11}$$

The chain rule gives

$$P_\varepsilon \mathbf{u}_\varepsilon = \varepsilon^{-1}\rho_+(\xi_3)\frac{\partial^2 \mathbf{u}_\varepsilon}{\partial t^2} - \varepsilon^{-3}\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(A_{ij}(\xi_3)\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right)$$
$$-\varepsilon^{-4}\sum_{i,j=1}^{3}\frac{\partial}{\partial x_i}\left(A_{ij}(\xi_3)\frac{\partial \mathbf{u}_\varepsilon}{\partial \xi_j}\right) - \varepsilon^{-4}\sum_{i,j=1}^{3}\frac{\partial}{\partial \xi_i}\left(A_{ij}(\xi_3)\frac{\partial \mathbf{u}_\varepsilon}{\partial x_j}\right)$$
$$-\varepsilon^{-5}\sum_{i,j=1}^{3}\frac{\partial}{\partial \xi_i}\left(A_{ij}(\xi_3)\frac{\partial \mathbf{u}_\varepsilon}{\partial \xi_j}\right).$$

$$\boxed{\mathbf{u}_\varepsilon \Rightarrow \mathbf{u}_\varepsilon^{(J)}}$$

Asymptotic analysis of the problem

Solution

$$
\begin{aligned}
\mathbf{u}_\varepsilon{}^{(J)}(x_1, x_2, x_3, t) &= \sum_{q+l=0}^{J} \varepsilon^{q+l} \sum_{\substack{i:|i|=l\\ i_j\in\{1,2\}}} N_{q,\, \mathbf{i_1}\dots i_l}\left(\frac{x_3}{\varepsilon}\right) \frac{\partial^{q+l}\mathbf{w}_\varepsilon{}^{(J)}(x_1, x_2, t)}{\partial t^q \partial x_{i_1}\dots\partial x_{i_l}} \\
&+ \sum_{q+l=0}^{J} \varepsilon^{q+l+2} \sum_{\substack{i:|i|=l\\ i_j\in\{1,2\}}} M_{q,\, \mathbf{i_1}\dots i_l}\left(\frac{x_3}{\varepsilon}\right) \frac{\partial^{q+l}\psi_\varepsilon{}^{(J)}(x_1, x_2, t)}{\partial t^q \partial x_{i_1}\dots\partial x_{i_l}} \quad \text{in } D_\varepsilon^+ \times (0, T),
\end{aligned}
$$

$$
\mathbf{v}_\varepsilon^{(J)}(x, t) = \sum_{k=0}^{J} \varepsilon^k \mathbf{v}_k(x, t), \quad p_\varepsilon^{(J)}(x, t) = \sum_{k=0}^{J} \varepsilon^k p_k(x, t) \text{ in } D^- \times (0, T),
$$

$$
\psi_\varepsilon^{(J)}(x_1, x_2, t) = 2\tilde{\nu} D(\mathbf{v}_\varepsilon^{(J)}(x_1, x_2, 0, t))\mathbf{e}_3 - p_\varepsilon^{(J)}(x_1, x_2, 0, t)\mathbf{e}_3,
$$

$$
\mathbf{w}_\varepsilon{}^{(J)}(x_1, x_2, t) = \sum_{k=0}^{J} \varepsilon^k \mathbf{w}_k(x_1, x_2, t), \quad (x_1, x_2, t) \in (0,1)^2 \times (0, T), \tag{12}
$$

with $\mathbf{w}_k$, $\mathbf{v}_k$, p_k 1-periodic in x_1, x_2, for any $k \in \{0, 1, ..., J\}$.

To determine the functions $\mathbf{w}_k$, $\mathbf{v}_k$, p_k we define the new function

$$\bar{\mathbf{w}}_\varepsilon^{(J)}(x_1,x_2,t) = \varepsilon^{-1}\left(\mathbf{w}_\varepsilon^{(J)}(x_1,x_2,t)\cdot\mathbf{e}_1\right)\mathbf{e}_1 + \varepsilon^{-1}\left(\mathbf{w}_\varepsilon^{(J)}(x_1,x_2,t)\cdot\mathbf{e}_2\right)\mathbf{e}_2 + \left(\mathbf{w}_\varepsilon^{(J)}(x_1,x_2,t)\cdot\mathbf{e}_3\right)\mathbf{e}_3. \tag{13}$$

We consider

$$\bar{\mathbf{w}}_\varepsilon^{(J)} = \sum_{k=-1}^{J} \varepsilon^k \bar{\mathbf{w}}_k(x_1,x_2,t), \tag{14}$$

so that it gives

$$\begin{cases} (w_k)_1 = (\bar{w}_{k-1})_1, \\ (w_k)_2 = (\bar{w}_{k-1})_2, \\ (w_k)_3 = (\bar{w}_k)_3 \quad \forall\, k \geq 0. \end{cases} \tag{15}$$

Asymptotic analysis of the problem

Leading term

The leading term of the asymptotic solution (12) is determined from

$$\begin{cases} \langle \rho_+ \rangle \dfrac{\partial^2 (w_0)_3}{\partial t^2} + \hat{\hat{J}} \Delta^2 (w_0)_3 - p_0 \Big/_{x_3=0} = g_3 & \text{in } (0,1)^2 \times (0,T), \\ \rho_- \dfrac{\partial \mathbf{v}_0}{\partial t} - \tilde{\nu} \Delta \mathbf{v}_0 + \nabla p_0 = \mathbf{f}, & \\ \mathrm{div}\ \mathbf{v}_0 = 0 & \text{in } D^- \times (0,T), \\ \mathbf{v}_0(x_1, x_2, -1, t) = \mathbf{0} & \text{in } (0,1)^2 \times (0,T), \\ \mathbf{v}_0(x_1, x_2, 0, t) = (c'_{-1})_1(t)\mathbf{e}_1 + (c'_{-1})_2(t)\mathbf{e}_2 + \dfrac{\partial (w_0)_3}{\partial t}(x_1, x_2, t)\mathbf{e}_3 & \\ \qquad \text{in } (0,1)^2 \times (0,T), & \\ (w_0)_3, \mathbf{v}_0, p_0 \ 1 - \text{periodic in } x_1, x_2, & \\ \mathbf{v}_0(x_1, x_2, x_3, 0) = \mathbf{0} \text{ in } D^-; (w_0)_3(x_1, x_2, 0) = \dfrac{\partial (w_0)_3}{\partial t}(x_1, x_2, 0) = 0 \text{ in } (0,1)^2, & \end{cases} \tag{16}$$

where $\hat{\hat{J}}$ is strictly positive constant, $\mathbf{c}_k = \langle \bar{\mathbf{w}}_k \rangle$, $\langle (w_0)_3 \rangle = 0$.

Asymptotic analysis of the problem

Discussion

For $k = -1$: $(c_{-1})_{1,2} = (\bar{w}_{-1})_{1,2}$, on the one hand, and $(w_0)_{1,2} = (\bar{w}_{-1})_{1,2}$, $(w_0)_3 = (\bar{w}_0)_3$, on the other hand. So the coupling condition $(16)_5$ can be written as

$$\mathbf{v}_0 = \frac{\partial \mathbf{w}_0}{\partial t} \quad \text{on } \Gamma_0. \tag{17}$$

Main points for construction of the functions $\mathbf{w}_k$, $\mathbf{v}_k$, p_k by induction:

- $(\bar{w}_{-1})_{1,2}$ with satisfy the system of elasticity theory with constant coefficients (homogenized coefficients) and the zero right-hand side:

$$\begin{cases} \hat{E}_1 \dfrac{\partial^2 (\bar{w}_{-1})_1}{\partial x_1^2} + \hat{E}_3 \dfrac{\partial^2 (\bar{w}_{-1})_2}{\partial x_1 \partial x_2} + \hat{E}_2 \dfrac{\partial^2 (\bar{w}_{-1})_1}{\partial x_2^2} = 0, \\ \hat{E}_2 \dfrac{\partial^2 (\bar{w}_{-1})_2}{\partial x_1^2} + \hat{E}_3 \dfrac{\partial^2 (\bar{w}_{-1})_1}{\partial x_1 \partial x_2} + \hat{E}_1 \dfrac{\partial^2 (\bar{w}_{-1})_2}{\partial x_2^2} = 0, \\ \bar{\mathbf{w}}_{-1} \ \text{1-periodic in } x_1, x_2, \end{cases} \tag{18}$$

Asymptotic analysis of the problem

Discussion (continuation)

- the triplet $((w_0)_3, \mathbf{v}_0, p_0)$ and $(\bar{w}_0)_{1,2}$ as a consequence,
- etc.
- the triplet $((w_k)_3, \mathbf{v}_k, p_k)$ is determined from the system like (16) with the coupling condition containing $\dfrac{\partial(\bar{w}_{k-1})_{1,2}}{\partial t}$,
- knowing the $(\bar{w}_{k-1})_{1,2}$ as the solution of the system like (18):

$$\begin{cases} \hat{E}_1 \dfrac{\partial^2(\bar{w}_k)_1}{\partial x_1^2} + \hat{E}_3 \dfrac{\partial^2(\bar{w}_k)_2}{\partial x_1 \partial x_2} + \hat{E}_2 \dfrac{\partial^2(\bar{w}_k)_1}{\partial x_2^2} = -(y_k)_1, \\ \hat{E}_2 \dfrac{\partial^2(\bar{w}_k)_2}{\partial x_1^2} + \hat{E}_3 \dfrac{\partial^2(\bar{w}_k)_1}{\partial x_1 \partial x_2} + \hat{E}_1 \dfrac{\partial^2(\bar{w}_k)_2}{\partial x_2^2} = -(y_k)_2, \\ \mathbf{y}_k = \hat{\hat{E}}_1 \nabla\big(\Delta(w_k)_3\big) + \mathbf{g}\delta_{k1} - \mathbf{R}_{k-1}, \\ \bar{\mathbf{w}}_k \ \text{1-periodic in } x_1, x_2, k > -1, \end{cases} \tag{19}$$

where the terms $\mathbf{R}_{k-1}$, depend on the functions $(\bar{w}_{k'-1})_1$, $(\bar{w}_{k'-1})_2$, $(w_{k'})_3$, $\mathbf{v}_{k'}$, $p_{k'}$ and their derivatives, with $k' \leq k-1$.

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Summary

Thank you for your attention!